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DISCUSSION OF LARGE HYPERBOLIC FUNCTIONS COMPUTED BY FISSION

(Published in December, 1950)

By John Sherman, K. L. Cooper, and F. T. Llewellyn

STRUCTURAL DIVISION

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DISCUSSION

JOHN SHERMAN,[9] M. ASCE.—It is correct that available tables for large hypes are arranged with intervals too large to allow interpolation to be made for intermediate values. The method to overcome this deficiency in the tables is based on the use of a trigonometric equation that has been in use for many years. Although the use of the word "fission" to designate this method is grammatically correct, it is open to question because it is based on the use of a long-established trigonometric equation. The writer objects to the author's use of the word "radian" to define a hyperbolic angle because radians are the only circular functions that are lacking from the close analogy between circular and hyperbolic functions.

The expression $\sinh v = \frac{1}{2}e^v$ is correct[10] to within one decimal place for values of v from 5.09 to 6.908 and to within two decimal places for values of v from 6.908 to 9.210. Therefore Eq. 10 gives precise results when v is larger than 5.09. This equation will give results correct to within 1% when v is larger than 3.00.

As a matter of fact, there is a suitable substitute for tables of large hypes. Every engineer has access to a log-log slide rule or to a four-place table of Naperian logarithms, either of which is sufficient equipment for the calculation of large hypes. The writer will present and demonstrate a method for testing the accuracy of values obtained by interpolating in tables.

Using inverse hyperbolic functions and denoting $\sinh v$ by u, then,

$$\sinh^{-1} u = \log_e (u + \sqrt{u^2 - 1}) \ldots\ldots\ldots\ldots\ldots\ldots (24a)$$

When the hyperbolic angle v is large, Eq. 24a is approximated by

$$\sinh^{-1} i = \log_e (2\,u) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24b)$$

which can be written in the form:

$$v = \log_e 2 \sinh v \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24c)$$

Let v be a small angle; then, for large angles Eq. 24c becomes

$$v \times 10^n = \log_e 2 \sinh v + \log_e 10^n \ldots\ldots\ldots\ldots\ldots (24d)$$

Recalling that $\log_e 10 = 2.302585$ Eq. 24d becomes

$$v \times 10^n = \log_e 2 \sinh v + 2.302585 \times n \ldots\ldots\ldots\ldots (25)$$

Conversely

$$\log_e 2 \sinh v = v \times 10^n - 2.302585 \times n \ldots\ldots\ldots\ldots (26)$$

Since $\log_e 2 \sinh v$ must be a positive number, it follows that the multiplier of 2.302585 in Eq. 26 must be the largest whole number of which its product

NOTE.—This paper by F. T. Llewellyn was published in December, 1950, as *Proceedings-Separate No. 49*. The numbering of footnotes and equations in this Separate is a continuation of the consecutive numbering used in the original paper.

[9] Designing Engr., Virginia Bridge Co., Roanoke, Va.

[10] *Hütte, Des Ingenieurs Taschenbuch*, Academischen Verein Hütte, Wilhelm Ernst und Sohn, Berlin, Vol. 1, 1902, p. 37.

2.302585, when subtracted from $v \times 10^n$, will leave a remainder. Since v is either known or to be determined, the multiplier 10^n can be omitted in Eqs. 25 and 26. Using the data from Example 1, a comparison of two solutions of Eq. 26 is as follows:

Description	Naperian log table	Log-log scale of slide rule
$v =$	28.49	28.49
$-12 \times 2.302585 =$	−27.631	−27.63
$\log_e 2 \sinh v =$	0.859	0.86
$2 \sinh 28.49 =$	2.36072×10^{12}	2.360×10^{12}
$\sinh 28.49 =$	$11{,}803.6 \times 10^8$	$11{,}800 \times 10^8$
Percentage error	0.0023	0.034

Using the data from Example 2, a similar comparison of two solutions of Eq. 25 is as follows:

Description	Naperian log table	Log-log scale of slide rule
$\log_e 1.426 =$	0.3549	0.355
$7 \times 2.302585 =$	+16.1181	+16.118
$v =$	16.4730	16.473

The foregoing results are the same as those obtained by the author.

There is no need for interpolation in the case of large hypes when Eqs. 12, 25, and 26 are used. Interpolation for large hypes is difficult for the following reasons: Consider an ordinate to the x-axis produced until it intersects the upper and lower branches of an equilateral hyperbola, which has its origin at the center of co-ordinates and its major axis on the x-axis. Draw a line from each point of intersection of the ordinate and of the curve to the origin. The area enclosed between these two radial lines and the curve is defined as the hyperbolic angle v. The hyperbolic sine, $\sinh v$, is equal to the length measured from the x-axis along the ordinate to the upper branch of the curve; and the hyperbolic cosine, $\cosh v$, is equal to the distance measured along the x-axis from the ordinate to the origin. The asymptotes of the hyperbola make an angle of 45° with the x-axis, and the $\sinh v$ and $\cosh v$ are equal when v is infinite. As v increases, the curve approaches the asymptote rapidly until, for successive values of $\cosh v$, the increase in area becomes very small. For very large values of v an infinitesimal increase in v causes the increase in $\sinh v$ and $\cosh v$ to become infinite. Therefore, if tables are to be adaptable to interpolation, the interval between successive tabulated values should diminish as v increases. Since this is not the case in any of the tables described by the author, these tables cannot be used for interpolation between large values of v.

Consider four successive terms, having equal intervals, taken from a table of hypes, and denote the first term by a. Let the successive differences between the four terms be d_1, d'_1, and d''_1; then, to find the value between the first two terms at the $\frac{1}{n}$th of the interval, $n\,d_1$ is added to a. This is the same as direct interpolation by proportional parts. Now let d_2 and d'_2 be the successive differences between d_1, d'_1, and d''_1, and let d_3 be the difference between d_2 and

d'_2; a better approximation for the interpolated term T is given by the following:

$$T = a + n\,d_1 + \frac{n}{2}(n-1)\,d_2 + \frac{n}{6}(n-1)(n-2)\,d_3 \ldots\ldots (27)$$

Eq. 27 can be used for interpolation for cases in which v is less than 5.09. For example, let sinh 4.75 be required when the values of sinh for 4.70, 4.80, 4.90, and 5.00 are known. The calculation is performed as follows:

Magnitude	4.70	4.80	4.90	5.00
sinh =	54.969	60.751	67.141	74.203
d_1 =		5.782	6.390	7.062
d_2 =			0.608	0.672
d_3 =				0.064

$\sinh 4.75 = 54.969 + \frac{5.782}{2} - \frac{0.608}{8} + \frac{0.064}{16} = 57.788$, which is the correct value. Interpolation by proportional parts gives a value $54.969 + 2.891 = 57.860$, which is 0.128% in error.

When Eq. 26 is applied to determine sinh 28.49 from the tabulated values of sinh 28, sinh 29, sinh 30, and sinh 31, it is found that d_2 is larger than d_1, and d_3 is larger than d_2; thus it is evident that interpolation is not possible. Calculating values of sinh for 28.4, 28.5, 28.6, and 28.7, Eq. 27 gives a value of $11{,}804.6 \times 10^8$ for sinh 28.49, which is correct to within 0.075%. In general it can be stated that interpolation can be made by proportional parts when d_2 approaches zero and that Eq. 27 will give precise results when each difference is about one tenth of the preceeding difference. When the differences approach equality, quantities cannot be interpolated without decreasing the interval.

Summary.—This discussion is offered to demonstrate that it is not necessary to have tables for the purpose of calculating large hypes. The author's method is excellent when one has access to the Hayashi tables.[11] Otherwise the procedure in this discussion will give results sufficiently precise. The writer is of the opinion that Eq. 27 is a much more workable test for interpolation than Eq. 6 and that hyperbolic functions are not indispensable. The most accurate method of computing cubic equations is that proposed by W. G. Horner, which can be used for the solution of quartic equations after the resolvent cubic is obtained. With the theorem of J. C. Sturm used to obtain the nature of the roots, the solution by the Horner method is an elegant application of mathematics. The solution of cubic and quartic equations by radicals, which involve the use of circular and hyperbolic functions, is more expedient than the Horner method and is sufficiently precise. However, when they find that the solution of a problem by a direct method involves hyperbolic functions, many engineers will use one of the longer methods, which can be worked by simple arithmetic, until they find that hyperbolic functions can be computed readily and precisely. The writer, therefore, believes that the paper is timely and important.

[11] "Fünfstellige Tafeln der Kreis, und Hyper belfunktionen," by K. Hayashi, Walter de Gruyter and Co., Berlin, 1921, p. 175.

K. L. Cooper.[12]—For the solution of cubic and quartic equations, the author has found the need for hyperbolic functions of arguments larger than six radians. From a survey and analysis of available reference tables of natural functions he has concluded that interpolation by proportional parts may lead to excessive error for large arguments, and, therefore, that either more comprehensive tables or a better method must be sought.

More comprehensive tables are available.[13] The tables prepared by L. J. Comrie make it possible to obtain the values for the author's examples by proportional parts, although this is not necessarily either the best or the shortest method.

His examples will serve to illustrate the use of these tables.

Example 1.—To compute sinh 28.49, from the Comrie tables find $e^{28.4} = 2.157562 \times 10^{12}$ and $e^{28.5} = 2.384475 \times 10^{12}$. Hence, sinh $28.49 = \frac{1}{2} \times 2.3618 \times 10^{12}$ or 1.1809×10^{12}.

Example 2.—To compute arcosh $(713 \times 10^4) = y$, we have $2 \cosh y = e^y = 1{,}426 \times 10^4$. Then, $e^{16.5} = 1.465072 \times 10^7$ and $e^{16.4} = 1.325652 \times 10^7$. Hence $y = 16.472$.

In the explanation to his tables, Mr. Comrie says[14]

> "* * * there are several ways in which values for non-tabular arguments may be obtained. Of these the best is probably that in which x is split into two parts a (a tabular argument) and b, so that $e^x = e^{a+b} = e^a \times e^b$."

This is the method to which the author draws attention as being somewhat less laborious and much more accurate than the use of proportional parts. It is not necessary, however, that a should be large and b should be small, as the author suggests, but only that b should fall into such tables as give sufficient decimal places. Thus, when the argument x is even, it will often be possible to reduce the author's two tabular references to one, and, when it is odd, it will be possible to reduce them to one entry for two adjacent values. For the inverse operation the given value is split into two equal parts by taking the square root. This procedure is illustrated by the same examples:

Example 3.—If $a = 14.25$ and $b = 14.24$ (both listed in Comrie's tables), then sinh $28.49 = \frac{1}{2} \times 1.544 \times 1.529 \times 10^{12} = 1.180 \times 10^{12}$.

Example 4.—If $a = b = (2 \times 713 \times 10^4)^{\frac{1}{2}} = 37.8 \times 10^2$ from tables $e^{8.23} = 3{,}752$ and $e^{8.24} = 3{,}790$. Hence $y = 2 \times 8.237 = 16.47$.

It should not be overlooked that repeated fission will make it possible to obtain the values of the hyperbolic functions of large arguments from tables that are restricted to small arguments.

Example 5.—Sinh $28.49 = \frac{1}{2} (e^6)^4 \times e^{2.25} \times e^{2.24}$ or $(e^{4.07})^7$.

Although the author has emphasized the use of natural functions, his problem may be solved without them. If the required accuracy is 1%, the

[12] Principal Research Officer, Div. of Forest Products, Commonwealth Scientific and Industrial Research Organization, Melbourne, Australia.

[13] "Chambers's Six-Figure Mathematical Tables," by L. J. Comrie, W. & R. Chambers, Ltd., London, Vol. II, 1949.

[14] *Ibid.*, p. xvi.

following simple approximations are satisfactory for values of x from 0 to 3:

$$\frac{\sinh x}{x} = 1 + \frac{0.947\,x^2}{6.65 - x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28a)$$

$$\cosh x = 1 + \frac{2.23\,x^2}{5.2 - x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28b)$$

These expressions combined will also give $\tanh x$ to the same degree of accuracy. At $x = 3$ the error involved in taking $\sinh x = \cosh x = \frac{1}{2}\,e^x$ is 0.25%, and in taking $\tanh x = 1$ the error is 0.5%. For values of x larger than 3, logarithms can be used to evaluate $\frac{1}{2}\,e^x$.

Example 6.—To compute $x = \sinh 28.49 = \frac{1}{2}\,e^{28.49}$ we have $\ln 2\,x = 28.49$. From an elementary table of logarithms $\ln 10^{12} = 27.6310$ and $\ln 2.361 = 0.8590$. Therefore, $x = \frac{1}{2} \times 2.361 \times 10^{12}$ or 1.180×10^{12}. The same solution will be obtained using common logarithms: $\log 2\,x = 28.49 \times 0.4343 = 12.3732$. Therefore $x = \frac{1}{2} \times 2.361 \times 10^{12}$.

For the inverse operation logarithms can be used for any values because

$$\operatorname{arsinh} x = \ln (x + \sqrt{x^2 + 1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (29a)$$

$$\operatorname{arcosh} x = \ln (x \pm \sqrt{x^2 - 1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (29b)$$

For $x = 10$, the errors involved in the approximations $\operatorname{arsinh} x = \operatorname{arcosh} x = \ln 2\,x$ are about 0.25%.

Example 7.—$y = \operatorname{arcosh} (713 \times 10^4) = \ln (2 \times 713 \times 10^4) = 2.3026$ (log $1{,}426 \times 10^4$) = 16.47. This value may be either positive or negative—a point not brought out in the paper.

A slide rule and an elementary table of logarithms suffice to solve the author's problem. With the additional aid of the writer's two simple approximations, any values of the hyperbolic functions or the inverse hyperbolic functions can be obtained to an accuracy better than 1%.

F. T. LLEWELLYN,[15] M. ASCE.—In appraising the efficiency of any mathematical formula it is helpful to consider its two phases, namely, solution and computation. The solution phase deals with algebraic forms of expression, whereas the computation phase comprises the finding of numerical values therefrom. It will be found, then, that the most "elegant" solution may not, and very often does not, lead to the most efficient method of computation.

Furthermore, in the field at issue, the author affirms that the practical need is not for additional research by mathematicians, for they have provided ample apparatus to choose from. Rather, the desideratum is to select and to standardize the fewest possible lines of procedure, so that they embrace the entire field and are arranged so that they can be readily applied without repeated reference to the theorems of advanced mathematics. This was the approach

[15] Baton Rouge, La.

attempted in the paper, and upon this approach the writer's closure will be based.

The $y^{f(n)}$ type of series, to which the correction $(1 + n)$ pertains, is useful in those rare cases in which $y < 0.0001$, which is the smallest radian usually tabulated. The writer recently encountered two problems (one in industry and one in research) wherein such meticulosity was desirable. In those problems the series converged so rapidly that the third term (where $n = 2$) would have affected only the 19th decimal and, therefore, was disregarded. Thus, provided $y < 0.0001$, four partial expressions of this type are correct to at least 18 places, as follows:—

$$\cosh(y) \doteq 1 + \frac{y^2}{2} \quad \ldots\ldots (30a)$$

$$\cos(y) \doteq 1 - \frac{y^2}{2} \quad \ldots\ldots (30b)$$

$$\sinh(y) \doteq y + \frac{y^3}{6} \quad \ldots\ldots (30c)$$

$$\sin(y) \doteq y - \frac{y^3}{6} \quad \ldots\ldots (30d)$$

The complete expressions may be found in the Hayashi[16] tables.

In the case of larger radians, the several series converge more slowly, and many additional terms may be required until the labor of computation becomes prohibitive. For this reason, rather than the one suggested in the paper, convergent series of the $y^{f(n)}$ type are not suited to general use.

Another brief note, that may clarify Table 1, is the fact that all the values of R_E are expressible also in terms of the square of the pertinent interval. Thus, Cols. 1, 4, and 6 could read as follows:

Group	Interval v	Maximum error R_E (%)
1	0.0001	$12.5\ v^2$
2	0.001	$12.6\ v^2$
3	0.01	$12.7\ v^2$
4	0.1	$12.9\ v^2$
5	1	$13.1\ v^2$
6	5	$41.0\ v^2$
7	10	$809.0\ v^2$

In the matter of terminology Mr. Sherman objects to the writer's use of the words fission and radian. Apparently Mr. Sherman (but not Mr. Cooper) accepts the abbreviations "hype" and "trig." The following explanation will show that the terms were not chosen idly.

The word "fission" was adopted as being more terse than the term "trigonometric equation" (which Mr. Sherman seems to favor), or than the term "addition theorem" (as used by certain other writers). "Fission" was preferred

[16] "Fünfstellige Tafeln der Kreis, und Hyperbelfunktionen," by K. Hayashi, Walter de Gruyter and Co., Berlin, 1921, pp. 178–179.

to "splitting," because it implies also a re-combination; it is shorter, and it dignifies the title of the paper.

The writer anticipated some adverse comment on the secondary meaning which he ventured to assign to the word "radian." He would point out, however, that many technical words are currently used in more than one sense. Thus, the word "natural" denotes Naperian as opposed to Briggs logarithms, and it also distinguishes numerical from logarithmic functions. Such words as "circular," "characteristic," and "argument" are also used in two senses. These double usages cause no confusion provided the context shows which meaning is intended. If the word "argument" had been substituted for "radian," it would have been necessary, in repeated instances, to add the phrase "in circular measure." The writer sought to avoid this needless verbiage by defining the two meanings of "radian" early in the paper.

To reciprocate, the writer would question Mr. Sherman's unqualified use of the word "interpolation" in cases in which he apparently means interpolation by PP. This is not a captious criticism because two methods of interpolation are in question. For small radians (in which $y < 6$, and the standard PP method is suitable), the corresponding hypes advance (within 13×10^{-4} %, or less) in arithmetical progression with the adjacent entries. For large radians (in which $y > 6$, and a fission method is suitable), the corresponding hypes advance (within 6×10^{-4} %. or less) in geometrical progression with the adjacent powers of e.

Based on his reference to Hütte,[10] Mr. Sherman finds two radian zones, extending from 5.09 to 9.21, within which the author's Eq. 10 would be correct to stated decimal places. This concept of overlapping zones is undoubtedly the more accurate, but, as stated in the paper, transition at ($x = 6$) to distinguish large from small radians, was a practical concession to the grouping and scope of current tables.

The writer has not had occasion to use Eq. 27. However, he agrees that, for small radians, it should give very close results, but its application must be quite laborious and its need almost nil, except perhaps in computing reference tables. By contrast, the author's two simple methods (PP when $y < 6$, and fission when $y > 6$) are sufficiently precise for the entire range of most structural problems.

In his summary, Mr. Sherman recommends the "indirect" Horner and Sturm theorems as accurate aids in computing cubic and quartic equations. But, in structural problems having cumbersome coefficients, the application of these "elegant" theorems would be much too laborious (the Sturm theorem especially) to suit the routine of busy designers. Moreover, each operation gives only one real root that may, or may not, be the root sought.

On the other hand, the "direct" methods cited in the paper (Cardan[2] for cubics, and preferably Euler[4] for quartics) give all the roots at a single operation. When these methods are used, labor is also reduced to a surprising extent by the adoption of standardized work-forms. A fuller discussion on the

[3] "Smithsonian Mathematical Tables," by G. F. Becker and C. E. Van Orstrand, Smithsonian Institution, Washington, D. C., 1909, p. 26.

[4] "Higher Algebra," by C. E. Fisher and I. J. Schwatt, Macmillan Co., New York, N. Y., 1924, p. 614.

various methods available for handling cubics and quartics would fall outside the purview of the paper.

Mr. Cooper's comments disclose the global reach of ASCE activities. Thus, the first hint to the writer as to the existence of the new Comrie tables[13] (in an English publication) has come via the antipodes. These tables are dated 1949, the year in which this paper was submitted to the ASCE. Had they been available at the time it was written, their reference would have formed a valuable addition to Table. 3.

It is no coincidence that, although working independently, Comrie used the same terms[14] as the present author in describing the splitting operation, because the basis for that operation had long been known. The writer has not seen a copy of the Comrie tables, and therefore he is not sure as to their spacing, but Mr. Cooper's examples imply that $v = 0.01$ at $x \doteq 14$, and that $v = 0.1$ at $x \doteq 28$, both intervals being closer than the spacing ($v = 1$) for the author's group 5 in either the Smithsonian or the Hayashi sets of tables. Thus, the Comrie ratios of error would here reduce to those of groups 3 or 4 (see Table 1, Cols. 1, 4, and 6), thereby affording still closer precision in the handling of large hypes.

It is to be noted, however, that Mr. Cooper's Examples 1 and 2, which were computed from the Comrie tables, entail major PP interpolation, whose scratch work he does not transcribe. The addition of this extra work would make the total labor somewhat greater than that of the author's corresponding examples, which were computed by fission at a single step, with only one easy mental interpolation in Example 2.

The possibility of computing large hypes by "repeated fission" (see Mr. Cooper's Example 5) was realized by the writer, but this method was not presented in the paper because it would be either more laborious or else less precise than the single and simple rule suggested in the section on "fission method."

The writer's more general reaction to Mr. Sherman's Eqs. 24 to 27, and to Mr. Cooper's Eqs. 28 to 29, may be combined. These contributors together present a total of six stimulating, but occasionally abstruse, formulas for computing hypes and for determining "individual" degrees of precision. Their results check quite well with those of the paper. Some of these formulas are new to the writer and others could be cited. This approach differs essentially from that of the paper, which was to standardize only two simple methods, one for large and one for small hypes (with a handy transition at $x = 6$), and to predetermine their limits of error "by groups." The author believes that the latter procedure, while perhaps less academic, is more helpful in the kind of problems encountered by structural engineers.

Both discussers envisage a free use of logs, which the author aimed to avoid in this field, primarily on account of the labor their use entails. In high precision work there is a further drawback to the use of log logs, in that the number of correct places in their mantissas is reduced by the maximum number of digits in the characteristic of the associated logs. This loss may become significant in cases in which the functions of very large hypes are sought.

In conclusion, the writer tenders his sincere thanks to Mr. Sherman and Mr. Cooper for their scholarly presentation of sidelights on the subject of his paper. He realizes that a few designers, relying upon their familiarity with advanced mathematics, may not need the specific rule that he has suggested as standard practice, and that such designers usually have their own preferred methods.

However, an appraisal of the discussions discloses no apparent reason to modify the hope (that is expressed by the paper and is supported by Mr. Sherman at the end of his summary) that the standardization of a fission method, with the aid of available tables, may be of practical value to numerous other designers who do not wish to disturb their routine work by a recurrent search for simple, yet adequate, means of computing large hypes.

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